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RESPONSE OF SLIGHTLY DAMPED GYROSCOPIC SYSTEMS,(U)
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RESPONSE OF SLIGHTLY DAMPED GYROSCOPIC SYSTEMS*

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Abstract

A second-order perturbation theory is developed for the response of slightly damped gyroscopic systems. The solution is based on the eigensolution for undamped gyroscopic systems and is expressed in terms of real quantities alone.

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1. Introduction

A general theory exists for the response of linear systems to arbitrary time-dependent excitations (Ref. 1). The solution is based on the so-called transition matrix. For high-order systems, the determination of the transition matrix is time-consuming, so that the method is not particularly attractive computationally. The situation is considerably better for special classes of systems, as shown in the following.

Undamped nongyroscopic systems are characterized by symmetric mass and stiffness matrices. The eigensolution for such systems consists of real eigenvalues and eigenvectors, where the latter are orthogonal with respect to the mass matrix. Taking advantage of these properties, the response can be obtained without much difficulty (Ref. 1). If viscous damping is present, then the eigensolution ceases to be real, even when the damping matrix is symmetric. Both the eigensolution and the response are significantly more difficult to obtain than for undamped systems

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(Ref. 1). A method for obtaining the eigensolution and response of undamped gyroscopic systems has been developed recently by the first author (Refs. 2 and 3). Taking advantage of the fact that the gyroscopic matrix is skew symmetric, the eigenvalue problem can be transformed into one in terms of symmetric matrices alone.

The problem of damped gyroscopic systems is considerably more complicated than those corresponding to the three special cases mentioned above. The sum of the damping matrix and the gyroscopic matrix is an arbitrary matrix. Hence, any advantage resulting from the symmetry or skew symmetry of coefficient matrices is lost, so that one must return to the general theory.

This paper is concerned with gyroscopic systems with small damping. In this case, damping can be regarded as a perturbation to the undamped gyroscopic system. Indeed, a second-order perturbation theory for the response of slightly damped gyroscopic systems is developed in this paper. The solution is based on the eigensolution for undamped gyroscopic systems and is expressed in terms of real quantities alone. The theory contains the case of slightly damped nongyroscopic systems (Ref. 4) as a special case.

2. Response of General Damped Gyroscopic Systems

The equations of motion of a general damped gyroscopic system can be written in the matrix form (Ref. 1)

$$M\ddot{\underset{\sim}{q}}(t) + (G + C)\dot{\underset{\sim}{q}}(t) + K\underset{\sim}{q}(t) = \underset{\sim}{Q}(t) \tag{1}$$

where M, C and K are n × n real symmetric matrices, G is an n × n real skew symmetric matrix, $\underset{\sim}{q}(t)$ is the n-dimensional configuration vector

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and $Q(t)$ is the associated n-dimensional force vector. We shall consider the case in which M and K are positive definite.

The solution of Eq. (1) can be most conveniently derived by transforming the equation to state form. To this end, we introduce the 2n-dimensional state vector and excitation vector

$$x(t) = [\dot{q}^T(t) \mid q^T(t)]^T \tag{2a}$$

$$X(t) = [Q^T(t) \mid 0^T]^T \tag{2b}$$

as well as the $2n \times 2n$ matrices

$$M^* = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & K \end{array}\right] \tag{3a}$$

$$K^* = \left[\begin{array}{c|c} G + C & K \\ \hline -K & 0 \end{array}\right] \tag{3b}$$

where M* is real symmetric and positive definite and K* is real but arbitrary. Definitions (2) and (3) permit us to rewrite Eq. (1) in the state form

$$M^*\dot{x}(t) + K^*x(t) = X(t) \tag{4}$$

Before discussing the solution of Eq. (4), we wish to reduce it to standard form, in which the coefficient matrix multiplying $\dot{x}$ is simply the identity matrix. Ordinarily, this would mean premultiplication of Eq. (4) by $(M^*)^{-1}$. Owing to the special nature of the problem, however, there exists a computationally superior procedure. Indeed, because the matrix M* is real symmetric and positive definite, we can use the Cholesky decomposition (Ref. 1) and write

$$M^* = LL^T \tag{5}$$

where L is a lower triangular matrix. Then, introducing the linear

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transformation

$$L^T\underset{\sim}{x}(t) = \underset{\sim}{u}(t) \quad , \quad \underset{\sim}{x}(t) = L^{-T}\underset{\sim}{u}(t) \tag{6}$$

where $L^{-T} = (L^T)^{-1} = (L^{-1})^T$, Eq. (4) can be reduced to the standard form

$$\dot{\underset{\sim}{u}}(t) = A\underset{\sim}{u}(t) + \underset{\sim}{U}(t) \tag{7}$$

in which

$$A = - L^{-1}K^*L^{-T} \tag{8}$$

is a real nonsymmetric matrix and

$$\underset{\sim}{U}(t) = L^{-1}\underset{\sim}{X}(t) \tag{9}$$

is a real vector.

The solution of Eq. (7) can be written in the general form involving the convolution integral (Ref. 1)

$$\underset{\sim}{u}(t) = \Phi(t)\underset{\sim}{u}(0) + \int_0^t \Phi(t,\tau)\underset{\sim}{U}(\tau)d\tau \tag{10}$$

where $\underset{\sim}{u}(0)$ is the initial vector and

$$\Phi(t,\tau) = e^{A(t-\tau)} \tag{11}$$

is known as the transition matrix.

The solution of Eq. (7) can also be obtained by modal analysis, which amounts to the determination of the Jordan form for A. To this end, let us consider the eigenvalue problem

$$A\underset{\sim}{u}_i = \lambda_i\underset{\sim}{u}_i \quad , \quad i=1,2,\ldots,2n \tag{12}$$

where λ_i and $\underset{\sim}{u}_i$ are the eigenvalues and eigenvectors of A. For simplicity, we shall assume that all the eigenvalues are distinct, so that the Jordan matrix is diagonal

$$\Lambda = \text{diag}[\lambda_i] \qquad (13)$$

The eigenvectors $\underset{\sim}{u}_i$ are known as the right eigenvectors of A and can be arranged in the square matrix

$$U = [\underset{\sim}{u}_1 \quad \underset{\sim}{u}_2 \quad \cdots \quad \underset{\sim}{u}_{2n}] \qquad (14)$$

The adjoint eigenvalue problem is

$$\underset{\sim}{v}_i^T A = \lambda_i \underset{\sim}{v}_i^T \quad , \quad i=1,2,\ldots,2n \qquad (15)$$

and has the same eigenvalues λ_i as before, as well as the left eigenvectors $\underset{\sim}{v}_i$ $(i=1,2,\ldots,2n)$. The left eigenvectors can be arranged in the square matrix

$$V = [\underset{\sim}{v}_1 \quad \underset{\sim}{v}_2 \quad \cdots \quad \underset{\sim}{v}_{2n}] \qquad (16)$$

The right eigenvectors are orthogonal to the left eigenvectors, a property known as biorthogonality. The eigenvectors can be normalized so as to satisfy

$$V^T U = U^T V = 2I \qquad (17)$$

where I is the identity matrix, in which case the Jordan matrix can be expressed as

$$V^T A U = 2\Lambda \qquad (18)$$

The reason for the factor 2 on the right side of Eqs. (17) and (18) will become evident later.

Because the vectors $\underset{\sim}{u}_i$ on the one hand and the vectors $\underset{\sim}{v}_i$ on the other hand are linearly independent, either set of vectors can be taken as a basis for a linear vector space L^{2n}, which implies that the solution of Eq. (7) can be represented as a linear combination of the eigenvectors

$\underset{\sim}{u}_i$ or of the eigenvectors $\underset{\sim}{v}_i$. Using the vectors $\underset{\sim}{u}_i$ as a basis, we can write

$$\underset{\sim}{u}(t) = \sum_{i=1}^{2n} \underset{\sim}{u}_i z_i(t) = U\underset{\sim}{z}(t) \tag{19}$$

where $\underset{\sim}{z}(t)$ is a 2n-vector with components $z_i(t)$. Introducing Eq. (19) into Eq. (7), premultiplying the result by V^T and considering Eqs. (17-18), we obtain

$$\dot{\underset{\sim}{z}}(t) = \Lambda\underset{\sim}{z}(t) + \underset{\sim}{Z}(t) \tag{20}$$

where

$$\underset{\sim}{Z}(t) = \frac{1}{2} V^T \underset{\sim}{U}(t) \tag{21}$$

Equation (20) represents a set of 2n independent equations having the solutions

$$z_i(t) = e^{\lambda_i t} z_i(0) + \int_0^t e^{\lambda_i (t-\tau)} Z_i(\tau) d\tau \quad , \quad i=1,2,\ldots,2n \tag{22}$$

or in vector form

$$\underset{\sim}{z}(t) = e^{\Lambda t} \underset{\sim}{z}(0) + \int_0^t e^{\Lambda(t-\tau)} \underset{\sim}{Z}(\tau) d\tau \tag{23}$$

where we note that $e^{\Lambda t}$ is a diagonal matrix. The formal solution is completed by first introducing Eq. (23) into Eq. (19) and then introducing the result into the second of Eqs. (6).

The interest lies in systems with a large number of degrees of freedom, measuring in the hundreds or more. For such systems, the procedures described above must receive closer scrutiny.

Solution (10) involves an integral containing the transition matrix Φ, Eq. (11). To generate the matrix Φ, it is necessary to expand a power series in A. Then, the solution is obtained by performing the indicated integration. For large-order systems, the process can be time-consuming. In addition, computer roundoff is likely to introduce computational errors. The other procedure, namely, that based on modal analysis, is also not very attractive for high-order systems. Indeed, the procedure requires the solution of the algebraic eigenvalue problem for the matrix A. But A, although real, is not symmetric. Hence, in general the eigenvalues and eigenvectors are likely to be complex quantities, as opposed to real quantities for real symmetric matrices. Moreover, computational algorithms for the eigensolution of arbitrary matrices are not nearly as efficient as those for real symmetric matrices. Hence, for high-order matrices, serious numerical difficulties can be encountered, and a different approach appears highly desirable.

3. Undamped Gyroscopic Systems

We observe from Eqs. (3) and (8) that in the absence of damping, $C = 0$, the matrix A becomes skew symmetric. In this case, the eigenvalues occur in pure imaginary complex conjugate pairs and the eigenvectors also occur in complex conjugate pairs, or

$$A\underset{\sim}{u}_i = \lambda_i \underset{\sim}{u}_i = i\omega_i \underset{\sim}{u}_i \quad , \quad i=1,2,\ldots,n \tag{24a}$$

$$A\bar{\underset{\sim}{u}}_i = \bar{\lambda}_i \bar{\underset{\sim}{u}}_i = -i\omega_i \bar{\underset{\sim}{u}}_i \quad , \quad i=1,2,\ldots,n \tag{24b}$$

where ω_i is the $\underline{i}$th natural frequency of oscillation. Because $A^T = -A$, the left eigenvectors of A are simply the complex conjugates $\bar{\underset{\sim}{u}}_i$. Indeed, if we write first

$$\underset{\sim}{v}_i^T A = \lambda_i \underset{\sim}{v}_i^T = i\omega_i \underset{\sim}{v}_i^T \quad , \quad i=1,2,\ldots,n \tag{25}$$

and then transpose the equations, we obtain

$$A\underset{\sim}{v}_i = -\lambda_i \underset{\sim}{v}_i = -i\omega_i \underset{\sim}{v}_i \quad , \quad i=1,2,\ldots,n \tag{26}$$

so that, comparing Eqs. (24b) and (26), we conclude that

$$\underset{\sim}{v}_i = \overline{\underset{\sim}{u}}_i \quad , \quad i=1,2,\dots,n \tag{27}$$

Hence, the 2n eigensolutions of A and A^T can be written in the special form of complex conjugate pairings, or

$$\begin{aligned} \lambda_i &= i\omega_i \quad , \quad \underset{\sim}{u}_i \quad , \quad \underset{\sim}{v}_i = \overline{\underset{\sim}{u}}_i \\ & \qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\dots,n \\ \overline{\lambda}_i &= -i\omega_i \quad , \quad \overline{\underset{\sim}{u}}_i \quad , \quad \overline{\underset{\sim}{v}}_i = \underset{\sim}{u}_i \end{aligned} \tag{28}$$

The above implies that it is not really necessary to solve the eigenvalue problem for A^T to obtain the left eigenvectors $\underset{\sim}{v}_i$.

Next, let us premultiply Eqs. (24) by -A and write

$$-A^2\underset{\sim}{u}_i = -i\omega_i A\underset{\sim}{u}_i = \omega_i^2\underset{\sim}{u}_i \quad , \quad i=1,2,\dots,n \tag{29a}$$

$$-A^2\overline{\underset{\sim}{u}}_i = i\omega_i A\overline{\underset{\sim}{u}}_i = \omega_i^2\overline{\underset{\sim}{u}}_i \quad , \quad i=1,2,\dots,n \tag{29b}$$

from which we conclude that the matrix $-A^2$ admits the same eigenvalue ω_i^2 for both eigenvectors $\underset{\sim}{u}_i$ and $\overline{\underset{\sim}{u}}_i$. Hence, the 2n eigenvalues of $-A^2$ consist of n eigenvalues ω_i^2 with double multiplicity. All the eigenvalues of $-A^2$ are real and positive, which is consistent with the fact that $-A^2$ is not only real and symmetric but also positive definite. But a real symmetric matrix is known to possess only real eigensolutions, so that it is not necessary to work with complex eigenvectors. Indeed, because any linear combination of the eigenvectors $\underset{\sim}{u}_i$ and $\overline{\underset{\sim}{u}}_i$ is also an eigenvector of $-A^2$ belonging to ω_i^2, we can simply choose the two real eigenvectors

$$\frac{1}{2}(\underset{\sim}{u}_i + \overline{\underset{\sim}{u}}_i) = \mathrm{Re}\, \underset{\sim}{u}_i = \underset{\sim}{y}_i \quad , \quad -\frac{i}{2}(\underset{\sim}{u}_i - \overline{\underset{\sim}{u}}_i) = \mathrm{Im}\, \underset{\sim}{u}_i = \underset{\sim}{z}_i \tag{30}$$

In conclusion, the eigenvalue problem for the undamped gyroscopic system, Eqs. (24), has been reduced to the eigenvalue problem for the real symmetric positive definite matrix $-A^2$. The latter problem has multiplicity two, with the real eigenvectors $\mathbf{y}_i$ and $\mathbf{z}_i$ belonging to the same eigenvalue ω_i^2 $(i=1,2,\ldots,n)$. These are essentially the results obtained in Ref. 2.

The reduction of the eigenvalue problem for an undamped gyroscopic system to that of a real symmetric positive definite matrix makes possible the use of a large variety of computationally efficient algorithms for the solution of the latter problem. The question remains, however, whether there is an efficient way of producing the response of a damped gyroscopic system. For arbitrarily large damping, the answer must be negative, as the addition of the matrix C to G destroys the skew symmetry of A and the symmetry and positive definiteness of $-A^2$. On the other hand, for relatively small damping, it is possible to devise a perturbation scheme taking advantage of results obtained for the undamped case. Before discussing the specific case of slightly damped gyroscopic systems, we shall present a second-order perturbation theory for arbitrary real matrices.

4. Second-Order Perturbation Theory for the Algebraic Eigenvalue Problem

Let us consider a $2n \times 2n$ arbitrary real matrix A. The eigenvalue problem for A is given by Eq. (12) and the adjoint eigenvalue problem is given by Eq. (15). The two eigenvalue problems are rewritten as

$$A\mathbf{u}_i = \lambda_i \mathbf{u}_i \quad , \quad i=1,2,\ldots,2n \tag{31a}$$

$$A^T\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad , \quad i=1,2,\ldots,2n \tag{31b}$$

where λ_i is the ith eigenvalue (the same for both problems) and $\mathbf{u}_i$ and $\mathbf{v}_i$ are eigenvectors of A and A^T, respectively. As mentioned in Sec. 2, $\mathbf{u}_i$ and $\mathbf{v}_i$ are called right and left eigenvectors of A, respectively, and possess the biorthogonality property. Upon the normalization indicated by Eqs. (17) and (18), the biorthonormality relations have the explicit form

$$\underset{\sim}{v}_j^T \underset{\sim}{u}_i = \underset{\sim}{u}_j^T \underset{\sim}{v}_i = 2\delta_{ij} \quad , \quad i,j=1,2,\ldots,2n \tag{32a}$$

$$\underset{\sim}{v}_j^T A \underset{\sim}{u}_i = \underset{\sim}{u}_j^T A^T \underset{\sim}{v}_i = 2\delta_{ij} \quad , \quad i,j=1,2,\ldots,2n \tag{32b}$$

It should be pointed out that eigenvalues and eigenvectors satisfying Eqs. (32) satisfy Eqs. (31) automatically, but the reverse is not necessarily true.

Next, let us assume that the matrix A can be expressed as

$$A = A_0 + A_1 \tag{33}$$

where A_0 and A_1 are known matrices, with the elements of A_1 being of one order of magnitude smaller than the elements of A_0. In fact, the matrix A can be regarded as the result of perturbing the matrix A_0 slightly. Consequently, we shall refer to A_0 as the unperturbed matrix, to A as the perturbed matrix and to A_1 as the perturbation matrix. The eigenvalue problem associated with A_0 will be identified as the unperturbed eigenvalue problem. Denoting the eigenvalues of A_0 by λ_{0i}, the right eigenvectors by $\underset{\sim}{u}_{0i}$ and the left eigenvectors by $\underset{\sim}{v}_{0j}$, and recognizing that they all satisfy eigenvalue problems of the type (31) and biorthonormality relations of the type (32), we can state by analogy the unperturbed eigenvalue problem

$$A_0 \underset{\sim}{u}_{0i} = \lambda_{0i} \underset{\sim}{u}_{0i} \quad , \quad i=1,2,\ldots,2n \tag{34a}$$

$$A_0^T \underset{\sim}{u}_{0i} = \lambda_{0i} \underset{\sim}{u}_{0i} \quad , \quad i=1,2,\ldots,2n \tag{34b}$$

and the corresponding biorthonormality relations

$$\underset{\sim}{u}_{0j}^{T}\underset{\sim}{u}_{0i} = \underset{\sim}{u}_{0j}^{T}\underset{\sim}{u}_{0i} = 2\delta_{ij} \quad , \quad i,j,=1,2,\ldots,2n \tag{35a}$$

$$\underset{\sim}{u}_{0j}^{T}A_0\underset{\sim}{u}_{0i} = \underset{\sim}{u}_{0j}^{T}A_0^{T}\underset{\sim}{u}_{0i} = 2\lambda_{0i}\delta_{ij} \quad , \quad i,j=1,2,\ldots,2n \tag{35b}$$

Now let us assume that unperturbed eigenvalues λ_{0i} and eigenvectors $\underset{\sim}{u}_{0i}$ and $\underset{\sim}{v}_{0i}$, satisfying Eqs. (34) and (35), are known. Then, our objective is to produce eigensolutions of the perturbed eigenvalue problem in terms of the unperturbed eigensolutions and the perturbation matrix A_1. To this end, we seek solutions of the perturbed eigenvalue problem in the form

$$\lambda_i = \lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \ldots \quad , \quad i=1,2,\ldots,2n \tag{36a}$$

$$\underset{\sim}{u}_i = \underset{\sim}{u}_{0i} + \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{2i} + \ldots \quad , \quad i=1,2,\ldots,2n \tag{36b}$$

$$\underset{\sim}{v}_i = \underset{\sim}{v}_{0i} + \underset{\sim}{v}_{1i} + \underset{\sim}{v}_{2i} + \ldots \quad , \quad i=1,2,\ldots,2n \tag{36c}$$

where the first subscript identifies the order of any particular term. For example, λ_{1i} is of order one, and hence is one order of magnitude smaller than λ_{0i} and one order of magnitude larger than λ_{2i}. Similarly, $\underset{\sim}{v}_{2i}$ is of order two, and hence is one order of magnitude smaller than $\underset{\sim}{v}_{1i}$ and two orders of magnitude smaller than $\underset{\sim}{v}_{0i}$. We note that the expansions (36) are not finite. Hence, if all terms in these expansions are determined, these expansions must converge to the eigensolutions of the perturbed eigenvalue problem. Because we shall determine and use only a finite number of terms, the resulting truncated expansions will provide approximate eigensolutions, which must tend to the unperturbed eigensolutions as A_1 tends to zero. Furthermore, as A_1 tends to zero the terms in any one of Eqs. (36) must maintain the same order of magnitude in relation to each other. For example, we cannot have the situation where $\underset{\sim}{u}_{2i}$ tends to zero as A_1 and $\underset{\sim}{u}_{1i}$ tends to zero as A_1^2.

The question arises as to what equations one should use to determine the perturbations to the eigenvalues and eigenvectors. Lancaster (Ref. 4), Franklin (Ref. 5) and Wilkinson (Ref. 6) present derivations in which Eqs. (31) are utilized. Franklin carries the derivation only through first order, while Lancaster and Wilkinson pursue the manipulations through second order. Wilkinson points out that normalization, as in Eqs. (32), is lost and that the perturbed eigenvectors must be renormalized (Ref. 6, Sec. 2.7). To avoid this difficulty, it is suggested in this paper that Eqs. (32) be used to determine the perturbations to the unperturbed eigensolutions. As mentioned previously, Eqs. (32) imply not only normalization but also satisfaction of the perturbed eigenvalue problem, Eqs. (31).

Substituting Eqs. (33) and (36) into Eqs. (32) and separating according to order of magnitude, we obtain

$$O(0):\quad \begin{aligned} \underset{\sim}{v}_{0j}^T \underset{\sim}{u}_{0i} &= 2\delta_{ij} \\ \underset{\sim}{v}_{0j}^T A_0 \underset{\sim}{u}_{0i} &= 2\lambda_{0i}\delta_{ij} \end{aligned} \qquad i,j=1,2,\ldots,2n \qquad (37a)$$

$$O(1):\quad \begin{aligned} \underset{\sim}{v}_{0j}^T \underset{\sim}{u}_{1i} + \underset{\sim}{v}_{1j}^T \underset{\sim}{u}_{0i} &= 0 \\ \underset{\sim}{v}_{0j}^T A_0 \underset{\sim}{u}_{1i} + \underset{\sim}{v}_{0j}^T A_1 \underset{\sim}{u}_{0i} + \underset{\sim}{v}_{1j}^T A_0 \underset{\sim}{u}_{0i} &= 2\lambda_{1i}\delta_{ij} \end{aligned} \qquad i,j=1,2,\ldots,2n \qquad (37b)$$

$$O(2):\quad \begin{aligned} &\underset{\sim}{v}_{0j}^T \underset{\sim}{u}_{2i} + \underset{\sim}{v}_{1j}^T \underset{\sim}{u}_{1i} + \underset{\sim}{v}_{2j}^T \underset{\sim}{u}_{0i} = 0 \\ &\underset{\sim}{v}_{0j}^T A_0 \underset{\sim}{u}_{2i} + \underset{\sim}{v}_{0j}^T A_1 \underset{\sim}{u}_{1i} + \underset{\sim}{v}_{1j}^T A_0 \underset{\sim}{u}_{1i} + \underset{\sim}{v}_{1j}^T A_1 \underset{\sim}{u}_{0i} \\ &\qquad + \underset{\sim}{v}_{2j}^T A_0 \underset{\sim}{u}_{0i} = 2\lambda_{2i}\delta_{ij} \end{aligned} \qquad i,j,=1,2,\ldots,2n \qquad (37c)$$

As expected, Eqs. (37a) are identical to Eqs. (35). Turning our attention to Eqs. (37b), we now wish to determine λ_{1i}, $\underset{\sim}{u}_{1i}$ and $\underset{\sim}{v}_{1i}$. Because $\underset{\sim}{u}_{1i}$ is a 2n-vector in the space L^{2n}, and because the vectors

$\underset{\sim}{u}_{0i}$ $(i=1,2,\ldots,2n)$ span this space, we can represent $\underset{\sim}{u}_{1i}$ as a linear combination of the $\underset{\sim}{u}_{0i}$. Hence we write

$$\underset{\sim}{u}_{1i} = \sum_{k=1}^{2n} \epsilon_{ik}\underset{\sim}{u}_{0k} \quad , \quad i=1,2,\ldots,2n \tag{38}$$

and we note that the smallness of $\underset{\sim}{u}_{1i}$ relative to $\underset{\sim}{u}_{0i}$ requires that the coefficients ϵ_{ik} be small. Following the same pattern, we represent $\underset{\sim}{v}_{1i}$ as a linear combination of the $\underset{\sim}{v}_{0i}$ as

$$\underset{\sim}{v}_{1i} = \sum_{k=1}^{2n} \gamma_{ik}\underset{\sim}{v}_{0k} \quad , \quad i=1,2,\ldots,2n \tag{39}$$

where the coefficients γ_{ik} are also small. Substituting Eqs. (38) and (39) into Eqs. (37b) and using Eqs. (37a), we obtain

$$\epsilon_{ij} + \gamma_{ji} = 0 \quad , \quad i,j=1,2,\ldots,n \tag{40a}$$

$$\lambda_{0j}\epsilon_{ij} + \lambda_{0i}\gamma_{ji} = -\frac{1}{2}\underset{\sim}{v}_{0j}^T A_1 \underset{\sim}{u}_{0i} + \lambda_{1i}\delta_{ij} \quad , \quad i,j=1,2,\ldots,2n \tag{40b}$$

When $i \neq j$, $\delta_{ij} = 0$, so that, solving for ϵ_{ij} and γ_{ji}, we obtain

$$\epsilon_{ij} = -\gamma_{ji} = \frac{\underset{\sim}{v}_{0j}^T A_1 \underset{\sim}{u}_{0i}}{2(\lambda_{0i}-\lambda_{0j})} \quad , \quad i,j=1,2,\ldots,2n, i\neq j \tag{41}$$

Alternatively, for $i = j$ Eqs. (40) become

$$\epsilon_{ii} + \gamma_{ii} = 0 \quad , \quad i=1,2,\ldots,2n \tag{42a}$$

$$\lambda_{0i}(\epsilon_{ii} + \gamma_{ii}) = -\frac{1}{2}\underset{\sim}{v}_{0i}^T A_1 \underset{\sim}{u}_{0i} + \lambda_{1i} \quad , \quad i=1,2,\ldots,2n \tag{42b}$$

Equations (42) yield the first order perturbations to the eigenvalues

$$\lambda_{1i} = \frac{1}{2}\underset{\sim}{v}_{0i}^T A_1 \underset{\sim}{u}_{0i} \quad , \quad i=1,2,\ldots,2n \tag{43}$$

Moreover, even though Eqs. (42) do not determine ϵ_{ii} and γ_{ii} uniquely, these equations are clearly satisfied if we take

$$e_{ii} = \gamma_{ii} = 0 \quad , \quad i=1,2,\ldots,2n \tag{44}$$

Equations (41), (43) and (44) determine the first-order corrections λ_{1i}, $\underset{\sim}{u}_{1i}$ and $\underset{\sim}{v}_{1i}$ fully. The reason for our assumption that the unperturbed eigenvalues be distinct is evident from Eqs. (41). It is easy to verify from Eqs. (41) and (43) that λ_{1i}, $\underset{\sim}{u}_{1i}$ and $\underset{\sim}{v}_{1i}$ do tend to zero as A_1 tends to zero.

Let us now consider Eqs. (37c) and obtain the second-order corrections. Proceeding as before, we let

$$\underset{\sim}{u}_{2i} = \sum_{k=1}^{2n} \tilde{e}_{ik}\underset{\sim}{u}_{0k} \quad , \quad i=1,2,\ldots,2n \tag{45a}$$

$$\underset{\sim}{v}_{2i} = \sum_{k=1}^{2n} \tilde{\gamma}_{ik}\underset{\sim}{v}_{0k} \quad , \quad i=1,2,\ldots,2n \tag{45b}$$

where $\tilde{e}_{ik}$ and $\tilde{\gamma}_{ik}$ are second-order quantities. Substituting Eqs. (45) into Eqs. (37c), and using the biorthonormality of the unperturbed eigensolutions, Eqs. (37a), we obtain

$$\tilde{e}_{ij} + \sum_{k=1}^{2n} e_{ik}\gamma_{jk} + \tilde{\gamma}_{ji} = 0 \quad , \quad i,j=1,2,\ldots,2n \tag{46a}$$

$$\lambda_{0j}\tilde{e}_{ij} + \frac{1}{2}\sum_{k=1}^{2n} e_{ik}\underset{\sim}{v}_{0j}^{T}A_1\underset{\sim}{u}_{0k} + \sum_{k=1}^{2n}\gamma_{0k}e_{ik}\gamma_{jk} + \frac{1}{2}\sum_{k=1}^{2n}\gamma_{jk}\underset{\sim}{v}_{0k}^{T}A_1\underset{\sim}{u}_{0i}$$

$$+ \lambda_{0i}\tilde{\gamma}_{ji} = \lambda_{2i}\delta_{ij} \quad , \quad i,j=1,2,\ldots,2n \tag{46b}$$

Using Eqs. (40), Eqs. (46) can be reduced to

$$\tilde{e}_{ij} + \tilde{\gamma}_{ji} = \sum_{k=1}^{2n} e_{ik}e_{kj} \quad , \quad i,j=1,2,\ldots,2n \tag{47a}$$

$$\lambda_{0j}\tilde{e}_{ij} + \lambda_{0i}\tilde{\gamma}_{ji} = (\lambda_{1i} - \lambda_{1j})e_{ij} + \lambda_{2i}\delta_{ij}$$

$$+ \sum_{k=1}^{2n}(\lambda_{0i} + \lambda_{0j} - \lambda_{0k})e_{ik}e_{kj} \quad , \quad i,j=1,2,\ldots,2n \tag{47b}$$

When $i \neq j$, $\delta_{ij} = 0$, so that Eqs. (47) can be solved for $\tilde{\epsilon}_{ij}$ and $\tilde{\gamma}_{ji}$, with the result

$$\tilde{\epsilon}_{ij} = \frac{1}{\lambda_{0i}-\lambda_{0j}} \left[(\lambda_{1j}-\lambda_{1i})\epsilon_{ij} + \sum_{k=1}^{2n} (\lambda_{0k}-\lambda_{0j})\epsilon_{ik}\epsilon_{kj} \right] , \quad i,j=1,2,\ldots,2n \tag{48a}$$

$$\tilde{\gamma}_{ji} = \frac{1}{\lambda_{0i}-\lambda_{0j}} \left[(\lambda_{1i}-\lambda_{1j})\epsilon_{ij} + \sum_{k=1}^{2n} (\lambda_{0i}-\lambda_{0k})\epsilon_{ik}\epsilon_{kj} \right] , \quad i,j=1,2,\ldots,2n \tag{48b}$$

When $i = j$, Eqs. (47) become

$$\tilde{\epsilon}_{ii} + \tilde{\gamma}_{ii} = \sum_{k=1}^{2n} \epsilon_{ik}\epsilon_{ki} , \quad i,j=1,2,\ldots,2n \tag{49a}$$

$$\lambda_{0i}(\tilde{\epsilon}_{ii} + \tilde{\gamma}_{ii}) = \lambda_{2i} + \sum_{k=1}^{2n} (2\lambda_{0i} - \lambda_{0k})\epsilon_{ik}\epsilon_{ki} , \quad i,j=1,2,\ldots,2n \tag{49b}$$

Solving Eqs. (49) for λ_{2i}, we obtain

$$\lambda_{2i} = \sum_{k=1}^{2n} (\lambda_{0k} - \lambda_{0i})\epsilon_{ik}\epsilon_{ki} \quad i=1,2,\ldots,2n \tag{50}$$

Although Eqs. (49) do not determine $\tilde{\epsilon}_{ii}$ and $\tilde{\gamma}_{ii}$ uniquely, the equations are satisfied if we take

$$\tilde{\epsilon}_{ii} = \tilde{\gamma}_{ii} = \frac{1}{2} \sum_{k=1}^{2n} \epsilon_{ik}\epsilon_{ki} , \quad i=1,2,\ldots,2n \tag{51}$$

Equations (48), (50) and (51) determine the second-order corrections, λ_{2i}, $\underset{\sim}{u}_{2i}$ and $\underset{\sim}{v}_{2i}$ fully. Recalling that λ_{1i}, ϵ_{ij} and γ_{ij} are proportional to A_1, we conclude that λ_{2i}, $\underset{\sim}{u}_{2i}$ and $\underset{\sim}{v}_{2i}$ are proportional to A_1^2, as anticipated.

5. Slightly Damped Gyroscopic (or Nongyroscopic) Systems

The second-order perturbation solution derived in the preceding section is quite general and the only restriction is that the matrix A be real. In the case of slightly damped gyroscopic (or nongyroscopic) systems, the matrix C can be regarded as being of one order of magnitude smaller than matrices G and K. Hence, recalling Eqs. (3b), (8), and (33), we can write

$$A_0 = -L^{-1}\left[\begin{array}{c|c} G & K \\ \hline -K & 0 \end{array}\right]L^{-T} \tag{52a}$$

$$A_1 = -L^{-1}\left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right]L^{-T} \tag{52b}$$

One should note that the matrix L can be partitioned, allowing a reduction in computational effort. Indeed, from Eqs. (3a) and (5), we can write

$$L = \left[\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array}\right] \tag{53}$$

where $M = L_1L_1^T$ and $K = L_2L_2^T$. Hence, Eqs. (52) can now be written as

$$A_0 = -\left[\begin{array}{c|c} L_1^{-1}GL_1^{-T} & L_1^{-1}L_2 \\ \hline -L_2^TL_1^{-T} & 0 \end{array}\right] \tag{54a}$$

$$A_1 = -\left[\begin{array}{c|c} L_1^{-1}CL_1^{-T} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{54b}$$

Note that $L_1^{-1}L_2$ is a lower triangular matrix, so that A_0 is banded The symmetry and skew-symmetry of A_1 and A_0, respectively, is readily apparent. Because matrix A_1 is of one order of magnitude smaller than A_0, the developments of the preceding section are readily applicable. In fact, because matrix A_0 is skew symmetric, the unperturbed eigensolutions of Sec. 4 are in reality the eigensolutions of the undamped gyroscopic system discussed in Sec. 3. Recalling the results of Sec. 4, we anticipate products of the form $\underset{\sim}{v}_0^T A_1 \underset{\sim}{u}_0$. Because only the first n rows and columns of the 2n × 2n matrix A_1 are nonzero, we need use only the upper halves of vectors $\underset{\sim}{v}_0$ and $\underset{\sim}{u}_0$.

Let us introduce primed subscripts where a prime indicates that the subscript in question has been increased by n. For example, $j' \equiv j + n$, $j = 1,2,\ldots,n$. Using this notation we have, from Eqs. (28), the pairings

$$\begin{aligned} &\lambda_{0i} = i\omega_{0i} \quad , \quad \underset{\sim}{u}_{0i} \quad , \quad \underset{\sim}{v}_{0i} = \overline{\underset{\sim}{u}}_{0i} \quad , \quad i=1,2,\ldots,n \\ &\lambda_{0i'} = -\,i\omega_{0i} \quad , \quad \underset{\sim}{u}_{0i'} = \overline{\underset{\sim}{u}}_{0i} \quad , \quad \underset{\sim}{v}_{0i'} = \overline{\underset{\sim}{v}}_{0i} = \underset{\sim}{u}_{0i} \quad , \\ &\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\ldots,n \end{aligned} \tag{55}$$

The first-order perturbation solution is obtained by introducing $\underset{\sim}{v}_{0i} = \overline{\underset{\sim}{u}}_{0i}$, $i = 1,2,\ldots,2n$ into Eqs. (41) and (43). The first-order eigenvalue perturbation is

$$\lambda_{1i} = \frac{1}{2}\, \underset{\sim}{u}_{0i}^H A_1 \underset{\sim}{u}_{0i} \quad , \quad i=1,2,\ldots,2n \tag{56}$$

where $\underset{\sim}{u}_{0i}^H \equiv \overline{\underset{\sim}{u}}_{0i}^T$. Because A_1 is real and symmetric, all λ_{1i} are real. Furthermore, assuming that the matrix C is positive semidefinite, A_1 is negative semidefinite and hence the λ_{1i} are nonpositive. Moreover, due to the real symmetric nature of the matrix A_1, we can restate Eqs. (56) as

$$\lambda_{1i} = \lambda_{1i'} = \frac{1}{2}\, \underset{\sim}{u}_{0i}^H A_1 \underset{\sim}{u}_{0i} \quad , \quad i=1,2,\ldots,n \tag{57}$$

which is to say that the first order perturbations to λ_{0i} and $\lambda_{0i'} = \overline{\lambda}_{0i}$ are the same. These familiar results are reassuring.

The coefficients ϵ_{ik} in the expansion (38) for $\underset{\sim}{u}_{1i}$ are

$$\epsilon_{ik} = \frac{\underset{\sim}{u}_{0k}^H A_1 \underset{\sim}{u}_{0i}}{2(\lambda_{0i} - \lambda_{0k})} \quad , \quad i,k=1,2,\ldots,2n, \; i \neq k \tag{58}$$

Recalling the symmetry of A_1 and the imaginary nature of the λ_0's, one can show that

$$\epsilon_{ik} = \bar{\epsilon}_{ki} \quad , \quad i,k=1,2,\ldots,2n \tag{59a}$$

$$\epsilon_{i'k'} = \bar{\epsilon}_{ik} \quad , \quad i,k=1,2,\ldots,n \tag{59b}$$

$$\epsilon_{ik'} = \bar{\epsilon}_{i'k} \quad , \quad i,k=1,2,\ldots,n \tag{59c}$$

Thus if we regard the $2n \times 2n$ array of ϵ's as partitioned into four $n \times n$ arrays, we need calculate only the upper triangular elements in the two upper $n \times n$ arrays. Furthermore, from Eqs. (58), we note that ϵ_{ik} and $\epsilon_{ik'}$ are related by the common vector $A_1\underset{\sim}{u}_{0i}$. Let us write Eqs. (38) as

$$\underset{\sim}{u}_{1i} = \sum_{k=1}^{n} (\epsilon_{ik}\underset{\sim}{u}_{0k} + \epsilon_{ik'}\bar{\underset{\sim}{u}}_{0k}) \quad , \quad i=1,2,\ldots,n \tag{60a}$$

Then, we can show that

$$\underset{\sim}{u}_{1i'} = \bar{\underset{\sim}{u}}_{1i} \quad , \quad i=1,2,\ldots,n \tag{60b}$$

and, from Eqs. (39), (41) and (59a), that

$$\underset{\sim}{v}_{1i} = -\bar{\underset{\sim}{u}}_{1i} \quad , \quad i=1,2,\ldots,2n \tag{61}$$

Next, let us denote the unperturbed eigenvectors as follows:

$$\underset{\sim}{u}_{0i} = \underset{\sim}{y}_{0i} + i\underset{\sim}{z}_{0i} \quad , \quad \underset{\sim}{u}_{0i'} = \underset{\sim}{y}_{0i} - i\underset{\sim}{z}_{0i} \quad , \quad i=1,2,\ldots,n \tag{62}$$

and note at this point that the normalization $\underset{\sim}{v}_{0i}^T\underset{\sim}{u}_{0i} = \underset{\sim}{u}_{0i}^H\underset{\sim}{u}_{0i} = 2$ can be realized by setting

$$\underset{\sim}{y}_{0i}^T\underset{\sim}{y}_{0i} = \underset{\sim}{z}_{0i}^T\underset{\sim}{z}_{0i} = 1 \quad , \quad i=1,2,\ldots,n \tag{63}$$

Inserting Eqs. (62) into Eqs. (43), we obtain

$$\left.\begin{matrix}\lambda_{1i} \\ \lambda_{1i'}\end{matrix}\right\} = \frac{1}{2}\left\{\begin{matrix}\underset{\sim}{u}_{0i}^{H} \\ \underset{\sim}{u}_{0i'}^{H}\end{matrix}\right\} A_1 \left\{\begin{matrix}\underset{\sim}{u}_{0i} \\ \underset{\sim}{u}_{0i'}\end{matrix}\right\} = \frac{1}{2}\left(\underset{\sim}{y}_{0i}^{T} A_1 \underset{\sim}{y}_{0i} + \underset{\sim}{z}_{0i}^{T} A_1 \underset{\sim}{z}_{0i}\right) \quad , \quad i=1,2,\ldots,n \qquad (64)$$

Because the matrix A_1 is negative semidefinite, Eqs. (64) confirm that all first-order perturbations to the eigenvalues are real and nonpositive. Moreover, inserting Eqs. (62) into Eqs. (58), we can write for $i \neq k$

$$\epsilon_{ik} = \frac{\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{z}_{0i} - \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{y}_{0i}\right) - i\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{y}_{0i} + \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{z}_{0i}\right)}{2(\omega_{0i} - \omega_{0k})}$$

$$1 \leq i, k \leq n \qquad (65a)$$

$$\epsilon_{ik} = \frac{\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{z}_{0i} + \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{y}_{0i}\right) - i\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{y}_{0i} - \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{z}_{0i}\right)}{2(\omega_{0i} + \omega_{0k})}$$

$$1 \leq i \leq n,\ n+1 \leq k \leq 2n \qquad (65b)$$

$$\epsilon_{ik} = \frac{\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{z}_{0i} + \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{y}_{0i}\right) + i\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{y}_{0i} - \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{z}_{0i}\right)}{2(\omega_{0i} + \omega_{0k})}$$

$$1 \leq k \leq n,\ n+1 \leq i \leq 2n \qquad (65c)$$

$$\epsilon_{ik} = \frac{\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{z}_{0i} - \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{y}_{0i}\right) + i\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{y}_{0i} + \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{z}_{0i}\right)}{2(\omega_{0i} - \omega_{0k})}$$

$$n+1 \leq i, k \leq 2n \qquad (65d)$$

Introducing Eqs. (62) and (65) into Eqs. (38), we obtain

$$\underset{\sim}{u}_{1i} = \sum_{\substack{k=1 \\ k\neq i}}^{n} \left[\frac{\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{z}_{0i} - \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{y}_{0i}\right) - i\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{y}_{0i} + \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{z}_{0i}\right)}{2(\omega_{0i} - \omega_{0k})} (\underset{\sim}{y}_{0k} + i\underset{\sim}{z}_{0k}) \right.$$

$$\left. + \frac{\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{z}_{0i} + \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{y}_{0i}\right) - i\left(\underset{\sim}{y}_{0k}^{T} A_1 \underset{\sim}{y}_{0i} - \underset{\sim}{z}_{0k}^{T} A_1 \underset{\sim}{z}_{0i}\right)}{2(\omega_{0i} + \omega_{0k})} (\underset{\sim}{y}_{0k} - i\underset{\sim}{z}_{0k}) \right]$$

$$i=1,2,\ldots,n \qquad (66a)$$

$$\underset{\sim}{u}_{1i'} = \overline{\underset{\sim}{u}}_{1i} \quad , \quad i=1,2,\ldots,n \tag{66b}$$

The second-order perturbation eigensolutions are obtained from Eqs. (48), (50) and (51). From Eqs. (50), (55) and (59a), we can write

$$\lambda_{2i} = i \sum_{k=1}^{n} [(\omega_{0k} - \omega_{0i}) e_{ik} \overline{e}_{ik} - (\omega_{0k} + \omega_{0i}) e_{ik'} \overline{e}_{ik'}] \qquad i=1,2,\ldots,n \tag{67a}$$

$$\lambda_{2i'} = \overline{\lambda}_{2i} \quad , \quad i=1,2,\ldots,n \tag{67b}$$

It should be evident from Eqs. (62) that λ_{2i} $(i = 1,2,\ldots,2n)$ is imaginary.

From Eqs. (51) and (59) we can write

$$\tilde{e}_{ii} = \tilde{\gamma}_{ii} = \frac{1}{2} \sum_{m=1}^{n} [e_{im} \overline{e}_{im} + e_{im'} \overline{e}_{im'}] \quad , \quad i=1,2,\ldots,n \tag{68a}$$

$$\tilde{e}_{i'i'} = \tilde{\gamma}_{i'i'} = \tilde{e}_{ii} \quad , \quad i=1,2,\ldots,n \tag{68b}$$

We note that $\tilde{e}_{ii}$ and $\tilde{\gamma}_{ii}$ $(i = 1,2,\ldots,n)$ are real. Comparing Eqs. (48a) and (48b), we can show that

$$\tilde{e}_{ik} = \overline{\tilde{\gamma}}_{ik} \quad , \quad i,k=1,2,\ldots,2n,\ i \neq k \tag{69}$$

This result, coupled with Eqs. (68), allows us to state that

$$\tilde{e}_{ik} = \overline{\tilde{\gamma}}_{ik} \quad , \quad i,k=1,2,\ldots,2n \tag{70}$$

and hence that

$$\underset{\sim}{v}_{2i} = \overline{\underset{\sim}{u}}_{2i} \quad , \quad i=1,2,\ldots,2n \tag{71}$$

Let us now write Eq. (48a) as

$$e_{ik} = \frac{1}{\lambda_{0i} - \lambda_{0k}} \left\{ (\lambda_{1k} - \lambda_{1i}) e_{ik} + \sum_{m=1}^{n} [(\lambda_{0m} - \lambda_{0k}) e_{im} e_{mk} - (\lambda_{0m} + \lambda_{0k}) e_{im'} e_{m'k}] \right\} \qquad i,k=1,2,\ldots,2n,\ i \neq k \tag{72}$$

One can now show that

$$\tilde{e}_{i'k'} = \bar{\tilde{e}}_{ik} \quad , \quad i,k=1,2,\ldots,n \tag{73a}$$

$$\tilde{e}_{ik'} = \bar{\tilde{e}}_{i'k} \quad , \quad i,k=1,2,\ldots,n \tag{73b}$$

From Eqs. (47a) and (70), we can write, in analogy with Eqs. (59a)

$$\tilde{e}_{ki} = -\bar{\tilde{e}}_{ik} + \sum_{m=1}^{n} (e_{km}e_{mi} + e_{km'}e_{m'i}) \qquad i,k=1,2,\ldots,2n, \; i \neq k \tag{73c}$$

Once again, if we regard the 2n × 2n array of e's as partitioned into four n × n arrays, we need calculate only the upper triangular elements in the upper two n × n arrays. Use of Eqs. (73) allow us to fill in the remaining elements. Let us write Eqs. (45a) as

$$\underset{\sim}{u}_{2i} = \sum_{k=1}^{n} (\tilde{e}_{ik}\underset{\sim}{u}_{0k} + \tilde{e}_{ik'}\underset{\sim}{u}_{0k'}) \quad , \quad i=1,2,\ldots,2n \tag{74}$$

Bearing in mind that $\underset{\sim}{u}_{0k'} = \bar{\underset{\sim}{u}}_{0k}$, Eqs. (73a), (73b) and (74) allow us to state that

$$\underset{\sim}{u}_{2i'} = \bar{\underset{\sim}{u}}_{2i} \quad , \quad i=1,2,\ldots,n \tag{75}$$

It may be convenient to exhibit the real and imaginary parts of the second-order perturbations explicitly. To this end, let us write

$$e_{ik} = e_{Rik} + ie_{Iik} \quad , \quad i,k=1,2,\ldots,2n \tag{76}$$

where subscripts R and I denote the respective real and imaginary parts of e_{ik}. Substitution of Eqs. (76) into Eqs. (67) yields

$$\lambda_{2i} = i\sum_{k=1}^{n}\left[\left(\omega_{0k} - \omega_{0i}\right)\left(e_{Rik}^2 + e_{Iik}^2\right) + \left(\omega_{0k} - \omega_{0i}\right)\left(e_{Rik'}^2 + e_{Iik'}^2\right)\right] \qquad i=1,2,\ldots,n \tag{77a}$$

$$\lambda_{2i'} = -\lambda_{2i} \quad , \quad i=1,2,\ldots,n \tag{77b}$$

Substituting Eqs. (76) into Eqs. (68), we obtain

$$\tilde{e}_{ii} = \frac{1}{2}\sum_{m=1}^{n}\left[\left(e_{Rim}^2 + e_{Iim}^2\right) + \left(e_{Rim'}^2 + e_{Iim'}^2\right)\right] \quad , \quad i=1,2,\ldots,n \tag{78a}$$

$$\tilde{e}_{i'i'} = \tilde{e}_{ii} \quad , \quad i=1,2,\ldots,n \tag{78b}$$

Using Eqs. (76), Eqs. (72) can be expressed as

$$\begin{aligned}\tilde{e}_{ik} = \frac{1}{\omega_{0i} - \omega_{0k}}\Big\{&(\lambda_{1k} - \lambda_{1i})(e_{Iik} - ie_{Rik}) \\ &+ \sum_{m=1}^{n}[(\omega_{0m} - \omega_{0k})(e_{Rim}e_{Rmk} - e_{Iim}e_{Imk}) \\ &\qquad - (\omega_{0m} + \omega_{0k})(e_{Rim'}e_{Rm'k} - e_{Iim'}e_{Im'k})] \\ &+ i\sum_{m=1}^{n}[(\omega_{0m} - \omega_{0k})(e_{Rim}e_{Imk} + e_{Iim}e_{Rmk}) \\ &\qquad - (\omega_{0m} + \omega_{0k})(e_{Rim'}e_{Im'k} + e_{Iim'}e_{Rm'k})]\Big\}\end{aligned}$$

$$i,k=1,2,\ldots,2n, \; i \neq k \tag{79}$$

Let us write

$$\tilde{e}_{ik} = \tilde{e}_{Rik} + i\tilde{e}_{Iik} \quad , \quad i,k=1,2,\ldots,2n \tag{80}$$

where $\tilde{e}_{Rik}$ and $\tilde{e}_{Iik}$ are the respective real and imaginary parts of $\tilde{e}_{ik}$. Equations (80) and (62) allow us to write Eqs. (45a) as

$$\begin{aligned}\underset{\sim}{u}_{2i} = \sum_{k=1}^{n}[&(\tilde{e}_{Rik} + \tilde{e}_{Rik'})\underset{\sim}{y}_{0k} + (-\tilde{e}_{Iik} + \tilde{e}_{Iik'})\underset{\sim}{z}_{0k} \\ &+ i(\tilde{e}_{Iik} + \tilde{e}_{Iik'})\underset{\sim}{y}_{0k} + i(\tilde{e}_{Rik} - \tilde{e}_{Rik'})\underset{\sim}{z}_{0k}]\end{aligned}$$

$$i=1,2,\ldots,n \tag{81a}$$

$$u_{2i'} = \bar{u}_{2i} \quad , \quad i=1,2,\ldots,n \tag{81b}$$

which completes the second-order perturbation solution.

It should be observed from the above that all the quantities needed for the evaluation of the first- and second-order perturbations are computed by means of the eigenvalues and eigenvectors of the unperturbed system, i.e., those of the undamped gyroscopic (or nongyroscopic) system.

6. System Response Based on the Second-Order Perturbation Eigensolution

In Sec. 2, we obtained the equations

$$z_i(t) = e^{\lambda_i t} z_i(0) + \int_0^t e^{\lambda_i(t-\tau)} Z_{1i}(\tau)d\tau \quad , \quad i=1,2,\ldots,2n \tag{82}$$

It will prove convenient to express Eqs. (82) in terms of the state vector and the associated excitation vector. From the results of Sec. 2, we can write

$$z_i(0) = \frac{1}{2} v_i^T L^T \begin{bmatrix} \dot{q}(0) \\ \hline q(0) \end{bmatrix} \quad , \quad i=1,2,\ldots,2n \tag{83}$$

$$Z_{1i}(t) = \frac{1}{2} v_i^T L^{-1} \begin{bmatrix} Q(t) \\ \hline 0 \end{bmatrix} \quad , \quad i=1,2,\ldots,2n \tag{84}$$

Equations (82) can now be written as

$$z_i(t) = e^{\lambda_i t} \frac{1}{2} v_i^T L^T \begin{bmatrix} \dot{q}(0) \\ \hline q(0) \end{bmatrix} + \int_0^t e^{\lambda_i(t-\tau)} \frac{1}{2} v_i^T L^{-1} \begin{bmatrix} Q(\tau) \\ \hline 0 \end{bmatrix} d\tau$$

$$i=1,2,\ldots,2n \tag{85}$$

which permits us to write the state vector

$$\begin{bmatrix} \dot{\underset{\sim}{q}}(t) \\ \hline \underset{\sim}{q}(t) \end{bmatrix} = L^{-T} \sum_{i=1}^{2n} \underset{\sim}{u}_i z_i(t) = \frac{1}{2} L^{-T} \sum_{i=1}^{2n} \left\{ e^{\lambda_i t} \underset{\sim}{u}_i \underset{\sim}{v}_i^T L^T \begin{bmatrix} \dot{\underset{\sim}{q}}(0) \\ \hline \underset{\sim}{q}(0) \end{bmatrix} \right.$$

$$\left. + \int_0^t e^{\lambda_i (t-\tau)} \underset{\sim}{u}_i \underset{\sim}{v}_i^T L^{-1} \begin{bmatrix} Q(\tau) \\ \hline \underset{\sim}{0} \end{bmatrix} d\tau \right\} \tag{86}$$

From Eq. (36a), we have

$$\lambda_i = \lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \dots \quad , \quad i=1,2,\dots,2n \tag{87}$$

In Sec. 5, we noted that λ_{0i} and λ_{2i} are imaginary quantities, while λ_{1i} is real and negative. Furthermore, we noted that

$$\begin{aligned} \lambda_{0i'} &= \overline{\lambda}_{0i} = -\lambda_{0i} \quad , \quad i=1,2,\dots,n \\ \lambda_{1i'} &= \lambda_{1i} \quad , \quad i=1,2,\dots,n \\ \lambda_{2i'} &= \overline{\lambda}_{2i} = -\lambda_{2i} \quad , \quad i=1,2,\dots,n \end{aligned} \tag{88}$$

where the meaning of a primed subscript was indicated in the previous section. In view of this, let us express Eqs. (87) as

$$\begin{aligned} \lambda_i &= -\gamma_i - i\omega_{di} \quad , \quad i=1,2,\dots,n \\ \lambda_{i'} &= \overline{\lambda}_i = -\gamma_i + i\omega_{di} \quad , \quad i=1,2,\dots,n \end{aligned} \tag{89}$$

where

$$\begin{aligned} \gamma_i &= -\lambda_{1i} \quad , \quad i=1,2,\dots,n \\ i\omega_{di} &= \lambda_{0i} + \lambda_{2i} \quad , \quad i=1,2,\dots,n \end{aligned} \tag{90}$$

and we note that the ω_{di} represent frequencies of damped oscillation. From Eqs. (36b), we have

$$\underset{\sim}{u}_i = \underset{\sim}{u}_{0i} + \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{2i} \quad , \quad i=1,2,\dots,2n \tag{91}$$

where we have shown that

$$\underset{\sim}{u}_{0i'} = \overline{\underset{\sim}{u}}_{0i} \quad , \quad i=1,2,\ldots,n$$

$$\underset{\sim}{u}_{1i'} = \overline{\underset{\sim}{u}}_{1i} \quad , \quad i=1,2,\ldots,n \tag{92}$$

$$\underset{\sim}{u}_{2i'} = \overline{\underset{\sim}{u}}_{2i} \quad , \quad i=1,2,\ldots,n$$

Hence, we can state that

$$\underset{\sim}{u}_{i'} = \overline{\underset{\sim}{u}}_{i} \quad , \quad i=1,2,\ldots,n \tag{93}$$

Furthermore, let us write

$$\underset{\sim}{u}_{i} = \sum_{k=1}^{2n} E_{ik}\underset{\sim}{u}_{0k} \quad , \quad i=1,2,\ldots,2n \tag{94}$$

where

$$E_{ik} = \delta_{ik} + e_{ik} + \tilde{e}_{ik} + \ldots \quad , \quad i,k=1,2,\ldots,2n \tag{95}$$

From Eqs. (36c), or

$$\underset{\sim}{v}_{i} = \underset{\sim}{v}_{0i} + \underset{\sim}{v}_{1i} + \underset{\sim}{v}_{2i} + \ldots \quad , \quad i=1,2,\ldots,2n \tag{96}$$

we can similarly show that

$$\underset{\sim}{v}_{i'} = \overline{\underset{\sim}{v}}_{i} \quad , \quad i=1,2,\ldots,n \tag{97}$$

and we can write

$$\underset{\sim}{v}_{i} = \sum_{k=1}^{2n} \overline{G}_{ik}\overline{\underset{\sim}{u}}_{0k} \quad , \quad i=1,2,\ldots,2n \tag{98}$$

where

$$G_{ik} \equiv \delta_{ik} - e_{ik} + \tilde{e}_{ik} + \ldots \quad , \quad i,k=1,2,\ldots,2n \tag{99}$$

Let us now reconsider Eq. (86). From Eqs. (89), (93) and (97) we conclude that the $\underline{i}$'th term is the complex conjugate of the $\underline{i}$th term, for every i and i'. Hence, we can rewrite Eq. (86) as

$$\begin{bmatrix} \dot{\mathbf{q}}(t) \\ \hline \mathbf{q}(t) \end{bmatrix} = L^{-T}\mathrm{Re} \sum_{i=1}^{n} \left\{ e^{\lambda_i t} \mathbf{u}_i \mathbf{v}_i^T L^T \begin{bmatrix} \dot{\mathbf{q}}(0) \\ \hline \mathbf{q}(0) \end{bmatrix} \right.$$

$$\left. + \int_0^t e^{\lambda_i (t-\tau)} \mathbf{u}_i \mathbf{v}_i^T L^{-1} \begin{bmatrix} \mathbf{Q}(\tau) \\ \hline \mathbf{0} \end{bmatrix} d\tau \right\} \tag{100}$$

From Eqs. (89), we can write

$$e^{\lambda_i t} = e^{-\gamma_i t}(\cos \omega_{di} t - i \sin \omega_{di} t) \quad , \quad i=1,2,\ldots,n \tag{101}$$

and from Eqs. (94) and (98), we have

$$\mathbf{u}_i \mathbf{v}_i^T = \sum_{k=1}^{2n} E_{ik} \mathbf{u}_{0k} \cdot \sum_{m=1}^{2n} \overline{G}_{im} \overline{\mathbf{u}}_{0m}^T \quad , \quad i=1,2,\ldots,2n \tag{102}$$

so that, using Eqs. (62), we can express Eqs. (102) as

$$\mathbf{u}_i \mathbf{v}_i^T = \sum_{k,m=1}^{n} \left[E_{ik}\left(\mathbf{y}_{0k} + i\mathbf{z}_{0k}\right) + E_{ik}\left(\mathbf{y}_{0k} - i\mathbf{z}_{0k}\right)\right]$$

$$\cdot \left[\overline{G}_{im}\left(\mathbf{y}_{0m}^T - i\mathbf{z}_{0m}^T\right) + \overline{G}_{im'}\left(\mathbf{y}_{0m}^T + i\mathbf{z}_{0m}^T\right)\right] \quad ,$$

$$i=1,2,\ldots,2n \tag{103}$$

Bearing in mind that the coefficients E_{ik} and $\overline{G}_{im}$ are complex quantities, we can write

$$\mathbf{u}_i \mathbf{v}_i^T = R_i + iI_i \quad , \quad i=1,2,\ldots,2n \tag{104}$$

where the $2n \times 2n$ matrices R_i and I_i are the respective real and imaginary parts of the $2n \times 2n$ matrix $\mathbf{u}_i \mathbf{v}_i^T$. Introducing Eqs. (101) and (104) into Eq. (100), we obtain

$$\begin{bmatrix} \dot{q}(t) \\ \hline q(t) \end{bmatrix} = L^{-T} \sum_{i=1}^{n} \left\{ \begin{aligned} & e^{-\gamma_i t} \cos \omega_{di} t \, R_i L^T \begin{bmatrix} \dot{q}(0) \\ \hline q(0) \end{bmatrix} \\ & + e^{-\gamma_i t} \sin \omega_{di} t \, I_i L^T \begin{bmatrix} \dot{q}(0) \\ \hline q(0) \end{bmatrix} \\ & + \int_0^t e^{-\gamma_i (t-\tau)} \cos \omega_{di}(t-\tau) R_i L^{-1} \begin{bmatrix} Q(\tau) \\ \hline 0 \end{bmatrix} d\tau \\ & + \int_0^t e^{-\gamma_i (t-\tau)} \sin \omega_{di}(t-\tau) I_i L^{-1} \begin{bmatrix} Q(\tau) \\ \hline 0 \end{bmatrix} d\tau \end{aligned} \right\} \tag{105}$$

which expresses the response in terms or real quantities alone.

Note that the above formulation remains valid in the case of non-gyroscopic systems, $G = 0$.

7. Numerical Examples

Consider the two-degree of freedom, damped gyroscopic system of Ref. 1 (shown here in Fig. 1). The equations of motion can be written in matrix form as

$$M\ddot{q}(t) + (G + C)\dot{q}(t) + Kq(t) = Q(t) \tag{106}$$

where

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad ; \quad G = \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \quad ; \quad K = \begin{bmatrix} k_1 - m\Omega^2 & 0 \\ 0 & k_2 - m\Omega^2 \end{bmatrix} \tag{107}$$

Let us take $m = 1\text{kg}$, $\Omega = 1\text{rad s}^{-1}$, $c = 0.1\text{kg s}^{-1}$, $k_1 = 3\text{kg s}^{-2}$, $k_2 = 4\text{kg s}^{-2}$, and note that k_1, k_2 and Ω have been picked such that matrix K is positive definite. From Eqs. (3) we form the matrices M* and K*

$$M^* \equiv \left[\begin{array}{c|c} m & 0 \\ \hline 0 & k \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$$K^* \equiv \left[\begin{array}{c|c} g+c & k \\ \hline -k & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -2 & 2 & 0 \\ 2 & 0 & 0 & 4 \\ \hline -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \end{array}\right] \tag{108}$$

The Cholesky decomposition matrix L, and its inverse, is

$$L = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 2 \end{array}\right] , \quad L^{-1} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/2 \end{array}\right] \tag{109}$$

where we recall the partitioned nature of matrix L. Following Eqs. (8) and (33) we can now form the matrices A_0 and A_1

$$A_0 \equiv -L^{-1}\left[\begin{array}{c|c} G & K \\ \hline -K & 0 \end{array}\right]L^{-T} = \left[\begin{array}{cc|cc} 0 & 2 & -\sqrt{2} & 0 \\ -2 & 0 & 0 & -2 \\ \hline \sqrt{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$$A_1 \equiv -L^{-1}\left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right]L^{-T} = \left[\begin{array}{cc|cc} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{110}$$

Computations of the perturbations to the undamped eigensolutions proceeds as indicated previously. The results are summarized in Table 1. One should note that the system eigenvalues and associated eigenvectors occur in complex conjugate pairs. Hence, we include only half of them. Furthermore, we include only the right eigenvectors.

Inspection of Table 1 reveals the quality of the convergence. It is also interesting to inspect convergence via a biorthonormality check of the perturbed eigenvectors. In Table 2, we present the matrix $\frac{1}{2}\underset{\sim}{V}^T\underset{\sim}{U}$. Numerical values within parentheses represent products

$$\frac{1}{2}\left(\underset{\sim}{v}_{0j}^T + \underset{\sim}{v}_{1j}^T + \underset{\sim}{v}_{2j}^T\right)\left(\underset{\sim}{u}_{0i} + \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{2i}\right) \quad , \quad i,j=1,2,\ldots,2n \tag{111a}$$

while values not within parentheses represent products

$$\frac{1}{2}\left(\underset{\sim}{v}_{0j}^T + \underset{\sim}{v}_{1j}^T\right)\left(\underset{\sim}{u}_{0i} + \underset{\sim}{u}_{1i}\right) \quad , \quad i,j=1,2,\ldots,2n \tag{111b}$$

The response to excitation in the form of the Dirac delta function, with impulse equal to 1 kgms^{-1} has been computed. Figure 2 represents a plot of the coordinate y(t) versus t due to an excitation applied in the y-direction. The two curves represent the response obtained by the general theory and the O(0) response. Within the accuracy of this plot, the O(0) + O(1) and O(0) + O(1) + O(2) responses are identical to that obtained by the general theory.

As another example, let us consider the slightly damped nongyroscopic system depicted in Fig. 3. The equations of motion considered are

$$\underset{\sim}{M}\ddot{\underset{\sim}{q}}(t) + \underset{\sim}{C}\dot{\underset{\sim}{q}}(t) + \underset{\sim}{K}\underset{\sim}{q}(t) = \underset{\sim}{Q}(t) \tag{112}$$

where

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \quad , \quad K = \begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix} \tag{113}$$

Matrices A_0 and A_1 are

$$A_0 = \left[\begin{array}{cc|cc} 0 & 0 & -\sqrt{5} & 0 \\ 0 & 0 & \sqrt{1.6} & -\sqrt{0.4} \\ \hline \sqrt{5} & -\sqrt{1.6} & 0 & 0 \\ 0 & \sqrt{0.4} & 0 & 0 \end{array}\right]$$

$$A_1 = \left[\begin{array}{cc|cc} -0.4 & \sqrt{0.02} & 0 & 0 \\ \sqrt{0.02} & -0.1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{114}$$

Summaries of the exact and perturbed eigensolutions are given in Table 3 and a biorthonormality check is given in Table 4. The quantities displayed in this table have the same meaning as those in Table 2. The response of the system to an excitation in the form $F_1(t) = 0$, $F_2(t) = \delta(t)$ is presented in Fig. 4. As in Fig. 2, the response obtained by the general theory and the O(0) + O(1) and O(0) + O(1) + O(2) responses are all given by one curve, whereas the O(0) response by the other curve.

Table 1

	EXACT	0(0)	0(0) + 0(1)	0(0) + 0(1) + 0(2)
λ_1	-0.018,894 -i10.936,368	-i10.936,426	-0.018,937 -i10.936,426	-0.018,937 -i10.936,369
λ_2	-0.031,058 -i13.019,857	-i13.020,448	-0.031,063 -i13.020,448	-0.031,063 -i13.019,857
$\underset{\sim}{u}_1$	[-0.001,883 -i10.615,661 0.368,951 -i10.012,786 0.929,520 +i10.015,959 0.011,363 +i10.788,277]	[-i10.615,412 0.369,048 0.929,410 i10.788,205]	[-0.001,881 -i10.615,412 0.369,048 -i10.012,782 0.929,410 +i10.015,954 0.011,360 +i10.788,205]	[-0.001,881 -i10.615,661 0.368,951 -i10.012,782 0.929,520 +i10.015,954 0.011,360 +i10.788,277]
$\underset{\sim}{u}_2$	[-0.014,992 -i10.788,156 -0.929,562 -i10.006.812 0.369,130 -i10.003,224 0.010,842 -i10.615,521]	[-i10.788,205 0.929,410 0.369,048 -i10.615,412]	[-0.014,987 -i10.788,205 -0.929,410 -i10.006,812 0.369,048 -i10.003,222 0.010,840 -i10.615,412]	[-0.014,987 -i10.788,156 -0.929,562 -i10.006,812 0.369,130 -i10.003,222 0.010,840 -i10.615.521]

Table 2

$$
\begin{bmatrix}
\begin{matrix} 0.999,724,755 \\ \begin{pmatrix} 1.000,000,044 \\ \quad +10.000,001,717 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,093,495 \\ \begin{pmatrix} 0.000,000,002 \\ \quad +10.000,000,196 \end{pmatrix} \end{matrix} &
\begin{matrix} 0.000,142,668 \\ \begin{pmatrix} -0.000,000,023 \\ \quad +10 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,057,828 \\ \begin{pmatrix} 0.000,000,022 \\ \quad -10.000,001,442 \end{pmatrix} \end{matrix} \\
 &
\begin{matrix} 0.999,800,548 \\ \begin{pmatrix} 1.000,000,022 \\ \quad +10.000,002,686 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,057,828 \\ \begin{pmatrix} 0.000,000,022 \\ \quad +10.000,001,442 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,142,668 \\ \begin{pmatrix} 0.000,000,008 \\ \quad -10 \end{pmatrix} \end{matrix} \\
 & &
\begin{matrix} 0.999,724,755 \\ \begin{pmatrix} 1.000,000,044 \\ \quad -10.000,001,717 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,093,495 \\ \begin{pmatrix} 0.000,000,002 \\ \quad -10.000,000,196 \end{pmatrix} \end{matrix} \\
 & & &
\begin{matrix} 0.999,800,548 \\ \begin{pmatrix} 1.000,000,022 \\ \quad -10.000,002,686 \end{pmatrix} \end{matrix}
\end{bmatrix}
$$

Table 3

	EXACT	0(0)	0(0) + 0(1)	0(0) + 0(1) + 0(2)
λ_1	-0.027,404 -i0.545,795	-i0.546,295	-0.027,391 -i0.546,295	-0.027,391 -i0.545,795
λ_2	-0.222,596 -i2.578,257	-i2.588,738	-0.222,609 -i2.588,738	-0.222,609 -i2.578,279
$\underset{\sim}{u}_1$	0.516,051 -i0.008,085 0.857,346 -i0.024,422 -0.029,793 +i0.125,764 -0.021,527 +i0.992,394	0.515,499 0.856,890 i0.125,942 i0.992,038	0.515,499 -i0.008,080 0.856,890 -i0.024,396 -0.029,732 +i0.125,942 -0.021,497 +i0.992,038	0.516,049 -i0.008,080 0.857,345 -i0.024,396 -0.029,732 +i0.125,764 -0.021,497 +i0.992,393
$\underset{\sim}{u}_2$	0.858,102 -i0.050,809 -0.515,320 -i0.000,788 -0.042,087 -i0.993,399 0.011,025 -i0.125,458	0.856,890 -0.515,499 i0.992,038 -i0.125,942	0.856,890 -i0.050,652 -0.515,499 -i0.000,790 -0.041,941 +i0.992,038 0.011,023 -i0.125,942	0.858,094 -i0.050,652 -0.515,321 -i0.000,790 -0.041,941 +i0.993,392 0.011,023 -i0.125,459

Table 4

$$
\begin{bmatrix}
\begin{matrix} 0.998,996,710 \\ \begin{pmatrix} 1.000,000,333 \\ -10.000,013,169 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,719,285 \\ \begin{pmatrix} 0.000,000,337 \\ -10.000,001,495 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,342,820 \\ \begin{pmatrix} 0.000,000,176 \\ +10 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,290,759 \\ \begin{pmatrix} 0.000,000,406 \\ -10.000,023,931 \end{pmatrix} \end{matrix} \\
 &
\begin{matrix} 0.997,776,602 \\ \begin{pmatrix} 1.000,001,775 \\ -10.000,009,643 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,290,759 \\ \begin{pmatrix} 0.000,000,406 \\ +10.000,023,931 \end{pmatrix} \end{matrix} &
\begin{matrix} 0.000,342,820 \\ \begin{pmatrix} -0.000,000,293 \\ +10 \end{pmatrix} \end{matrix} \\
 & &
\begin{matrix} 0.998,996,710 \\ \begin{pmatrix} 1.000,000,333 \\ +10.000,013,169 \end{pmatrix} \end{matrix} &
\begin{matrix} -0.000,719,285 \\ \begin{pmatrix} 0.000,000,337 \\ +10.000,001,495 \end{pmatrix} \end{matrix} \\
 & & &
\begin{matrix} 0.997,776,602 \\ \begin{pmatrix} 1.000,001,775 \\ +10.000,009,643 \end{pmatrix} \end{matrix}
\end{bmatrix}
$$

8. Summary and Conclusions

A second-order perturbation analysis was developed for the algebraic eigenvalue problem

$$(A_0 + A_1)\underset{\sim}{u}_i = \lambda_i \underset{\sim}{u}_i \quad , \quad i=1,2,\ldots,2n$$

where matrices A_0 and A_1 are real and where A_1 is one order of magnitude smaller than A_0. The analysis was based upon knowledge of the eigensolutions when A_1 is the null matrix. The perturbation theory was applied to slightly damped gyroscopic systems. In this case, A_0 is skew symmetric and A_1 is symmetric, so that special computational advantages can be realized. Note also that the nongyroscopic systems can be handled within the context of the same general formulation by simply letting $G = 0$.

As an example, a two-degree of freedom slightly damped gyroscopic system was analyzed. As another example, a nongyroscopic system has been treated. Even for relatively large damping, the perturbation results for both cases agree well with the solutions obtained by algorithms for general matrices. Because the present formulation is based on the eigensolution for real symmetric matrices, it should prove far superior for high-order systems.

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List of Figures

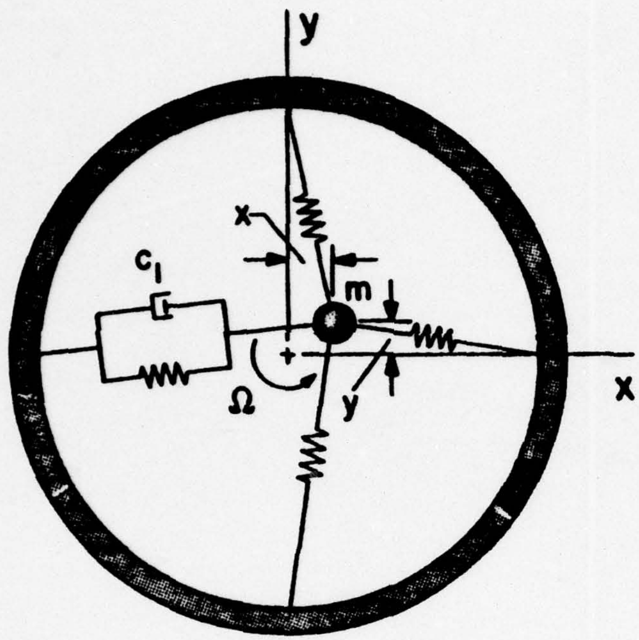


FIGURE I

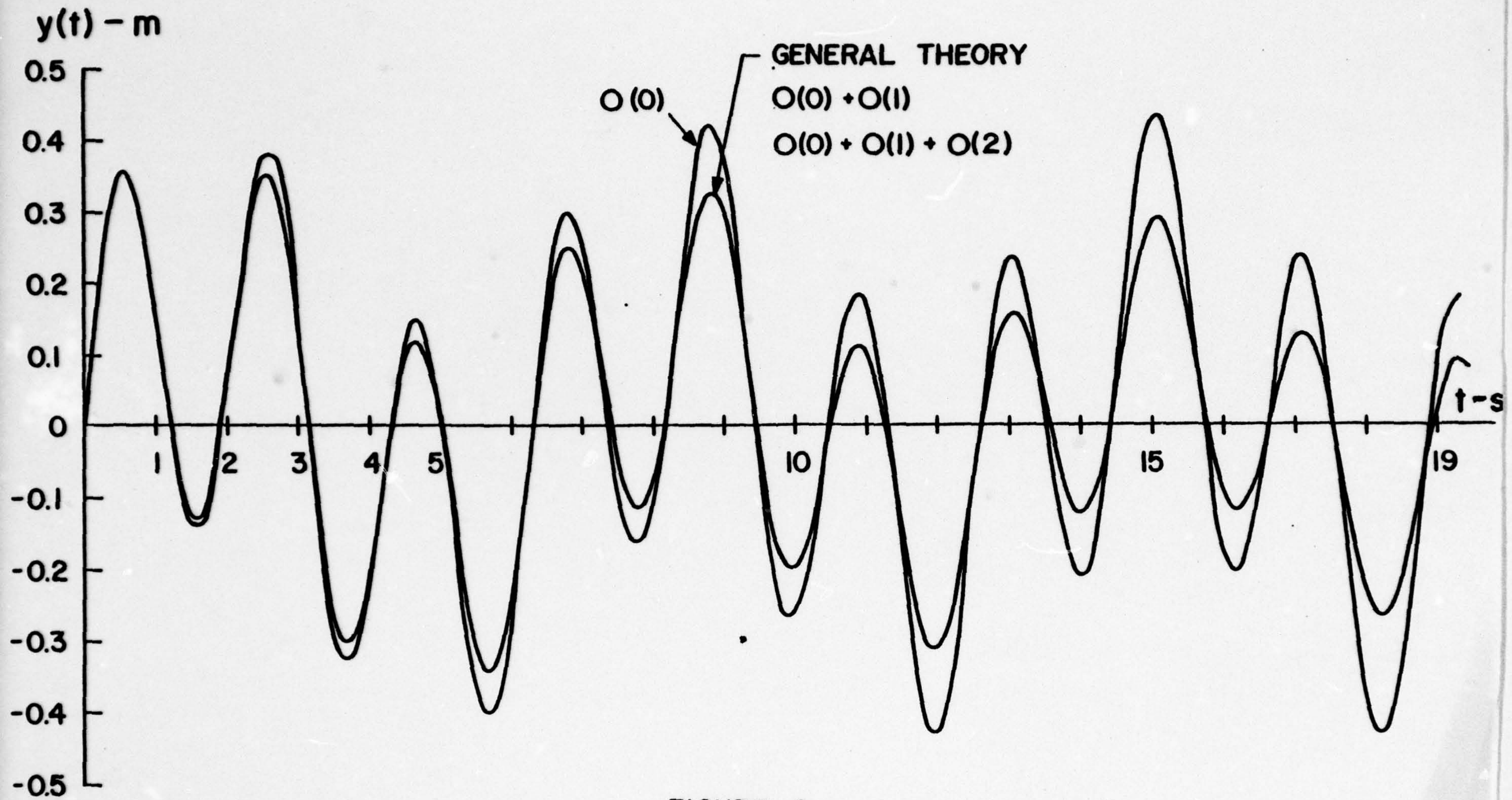

y(t) - m
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
1
2
3
4
5
10
15
19
t-s
O(0)
GENERAL THEORY
O(0) + O(1)
O(0) + O(1) + O(2)

FIGURE 2

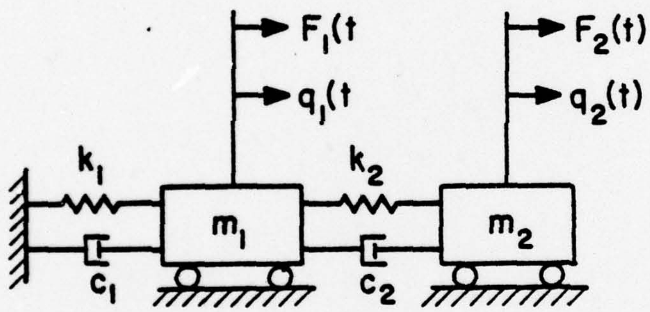
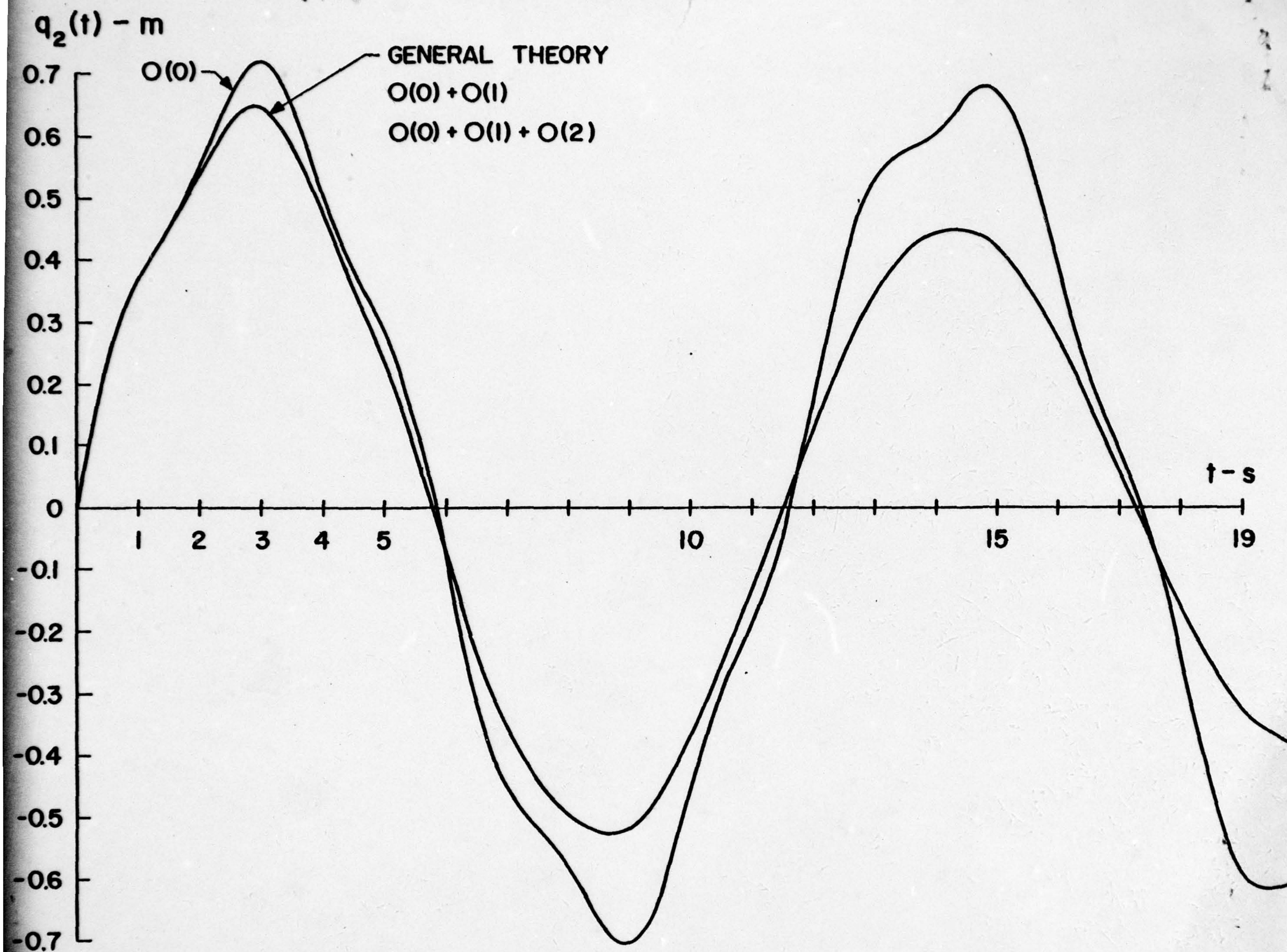


FIGURE 3



q2(t) - m
O(0)
GENERAL THEORY
O(0) + O(1)
O(0) + O(1) + O(2)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
1
2
3
4
5
10
15
19
t - s

FIGURE 4